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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 70/77 FEBRUARI

J. VAN DE LUNE

A DIVISIBILITY PROPERTY OF CERTAIN PRODUCTS
OF FACTORIALS

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A divisibility property of certain products of factorials

by

J. van de Lune

ABSTRACT

Recently RUDERMAN conjectured that for $n, a_1, a_2, \ldots, a_n \in \mathbb{N}$ the fraction

$$\frac{(na_1)!(na_2)! \ldots (na_n)!}{n^{n-1}(a_1!a_2!\ldots a_n!)^{n-1}(a_1+a_2+\ldots+a_n)!}$$

is actually an integer.
In this note it is shown that RUDERMAN's conjecture is correct.

KEY WORDS & PHRASES: *divisibility, factorials.*

In 1875 it was shown by BOURGUET (c.f.[1; p.63]) that if $n \in \mathbb{N}$, $n \geq 2$ and $a_1, a_2, \ldots, a_n$ are positive integers then the fraction

$$(1) \qquad \frac{(na_1)!(na_2)! \ldots (na_n)!}{a_1!a_2! \ldots a_n!(a_1+a_2+\ldots+a_n)!}$$

is actually an integer.

Recently it was conjectured by RUDERMAN (c.f. [2]) that if $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ are positive integers then the fraction

$$(2) \qquad \frac{(na_1)! \ldots (na_n)!}{n^{n-1}(a_1!\ldots a_n!)^{n-1}(a_1+\ldots+a_n)!}$$

is still an integer.

The main purpose of this note is to prove that RUDERMAN's conjecture is correct.

PROOF. The main tool in our proof is Legendre's well known factorization formula for the factorial

$$(3) \qquad m! = \prod_p p^{\sum_{i=1}^{\infty} \left[\frac{m}{p^i}\right]}, \quad (m \in \mathbb{N}).$$

Let p be any fixed prime.

CASE 1. $p \nmid n$

The exponent of p in the numerator of (2) is

$$(4) \qquad \sum_{k=1}^{n} \sum_{i=1}^{\infty} \left[\frac{na_k}{p^i}\right]$$

and the exponent of p in the denominator of (2) is

$$(5) \qquad (n-1) \sum_{k=1}^{n} \sum_{i=1}^{\infty} \left[\frac{a_k}{p^i}\right] + \sum_{i=1}^{\infty} \left[\frac{a_1+\ldots+a_n}{p^i}\right],$$

so that p is going to survive if (4) ≥ (5). Hence, it suffices to show that for all $i \in \mathbb{N}$

$$(6) \qquad \sum_{k=1}^{n} \left[\frac{na_k}{p^i}\right] \geq (n-1) \sum_{k=1}^{n} \left[\frac{a_k}{p^i}\right] + \left[\frac{a_1+\ldots+a_n}{p^i}\right].$$

Writing

$$(7) \qquad a_k = q_k p^i + r_k \quad \text{with} \quad 0 \leq r_k < p^i$$

we see that (6) may be written as

$$(8) \qquad \sum_{k=1}^{n} \left[nq_k + \frac{nr_k}{p^i}\right] \geq (n-1) \sum_{k=1}^{n} \left[q_k + \frac{r_k}{p^i}\right] + \left[q_1+\ldots+q_n + \frac{r_1+\ldots+r_n}{p^i}\right]$$

which is easily seen to be equivalent to

$$(9) \qquad \sum_{k=1}^{n} \left[\frac{nr_k}{p^i}\right] \geq \left[\frac{r_1+\ldots+r_n}{p^i}\right].$$

Defining

$$(10) \qquad R = \max\{r_1,\ldots,r_n\}$$

we have

$$(11) \qquad \sum_{k=1}^{n} \left[\frac{nr_k}{p^i}\right] \geq \left[\frac{nR}{p^i}\right] \geq \left[\frac{r_1+\ldots+r_n}{p^i}\right]$$

and the proof of case 1 is complete.

CASE 2. $p \mid n$.

We write $n = m \cdot p^{\alpha}$ with $(m,p) = 1$. The exponent of p in the numerator of (2) is again given by (4), whereas its exponent in the denominator is

$$(12) \qquad (n-1)\alpha + (n-1) \sum_{k=1}^{n} \sum_{i=1}^{\infty} \left[\frac{a_k}{p^i}\right] + \sum_{i=1}^{\infty} \left[\frac{a_1+\ldots+a_n}{p^i}\right],$$

so that p is going to survive if $(4) \geq (12)$.

Now observe that

$$(13) \qquad (4) = \sum_{k=1}^{n} \sum_{i=1}^{\infty} \left[\frac{mp^{\alpha} a_k}{p^i}\right] =$$

$$= \sum_{k=1}^{n} \left\{ mp^{\alpha-1} a_k + mp^{\alpha-2} a_k + \ldots + ma_k + \sum_{i=1}^{\infty} \left[\frac{ma_k}{p^i}\right] \right\} =$$

$$= \sum_{k=1}^{n} \left\{ ma_k \frac{p^{\alpha}-1}{p-1} + \sum_{i=1}^{\infty} \left[\frac{ma_k}{p^i}\right] \right\} \geq$$

(since $[mx] \geq m[x]$ for $x > 0$ and $m \in \mathbb{N}$)

$$\geq \frac{n-m}{p-1} \sum_{k=1}^{n} a_k + m \sum_{k=1}^{n} \sum_{i=1}^{\infty} \left[\frac{a_k}{p^i}\right].$$

Hence, it suffices to show that

$$(14) \qquad \frac{n-m}{p-1} \sum_{k=1}^{n} a_k \geq (n-1)\alpha + (n-m-1) \sum_{k=1}^{n} \sum_{i=1}^{\infty} \left[\frac{a_k}{p^i}\right] + \sum_{i=1}^{\infty} \left[\frac{a_1+\ldots+a_n}{p^i}\right].$$

For $k = 1,\ldots,n$ let $e(k)$ be the largest integer such that

$$(15a) \qquad \frac{a_k}{p^{e(k)}} \geq 1$$

and let e be the largest integer such that

$$(15b) \qquad \frac{a_1+\ldots+a_n}{p^e} \geq 1.$$

Then it suffices to show that

$$(16)\qquad \frac{n-m}{p-1}\sum_{k=1}^{n} a_k \geq (n-1)\alpha + (n-m-1)\sum_{k=1}^{n}\sum_{i=1}^{e(k)}\frac{a_k}{p^i} + \sum_{i=1}^{e}\frac{a_1+\ldots+a_n}{p^i}$$

$$(\text{since } p \mid n \Rightarrow p^\alpha > 1 \Rightarrow n > m \Rightarrow n-m-1 \geq 0)$$

or , equivalently, that

$$(17)\qquad \frac{n-m}{p-1}\sum_{k=1}^{n} a_k \geq (n-1)\alpha + (n-m-1)\sum_{k=1}^{n} a_k \frac{1-\frac{1}{p^{e(k)}}}{p-1} + \left(\sum_{k=1}^{n} a_k\right)\frac{1-\frac{1}{p^e}}{p-1}$$

which may be simplified to

$$(18)\qquad \frac{n-m-1}{p-1}\sum_{k=1}^{n}\frac{a_k}{p^{e(k)}} + \frac{1}{(p-1)p^e}\sum_{k=1}^{n} a_k \geq (n-1)\alpha.$$

From the definition of the numbers $e(k)$ and e it follows that it suffices to show that

$$(19)\qquad \frac{n-m-1}{p-1}\cdot n + \frac{1}{p-1} \geq (n-1)\alpha$$

or, equivalently, that

$$(20)\qquad n(n-m) \geq (n-1)\{1+\alpha(p-1)\}.$$

If $m=1$ we have by Bernoulli's inequality

$$(21)\qquad n = p^\alpha = \{1+(p-1)\}^\alpha \geq 1 + \alpha(p-1)$$

and (20) follows.

If $m \geq 2$ we observe that (20) may be rewritten as (note that $n = mp^{\alpha}$)

$$(22) \qquad \frac{n}{n-1}\, m \geq \frac{1+\alpha(p-1)}{p^{\alpha}-1}$$

Since

$$(23) \qquad \frac{n}{n-1}\, m > m \geq 2$$

and

$$(24) \qquad \frac{1+\alpha(p-1)}{p^{\alpha}-1} = \frac{1+\alpha(p-1)}{(p-1)(1+p+\ldots+p^{\alpha-1})} \leq \frac{1+\alpha(p-1)}{(p-1)\ \alpha} = 1 + \frac{1}{\alpha(p-1)} \leq 2$$

our proof is complete.

<u>REMARKS.</u> 1. From the above proof it also follows that if n is not a pure prime power (so that $m \geq 2$) then formula (20) holds true with $\geq$ replaced by $>$ so that the fraction

$$(25) \qquad \frac{(na_1)!\ldots(na_n)!}{\beta(n)n^{n-1}(a_1!\ldots a_n!)^{n-1}(a_1+\ldots+a_n)!}$$

is still an integer.

Here $\beta(n)$ is defined as follows: If $n = p_1^{e_1}\, p_2^{e_2}\ldots p_s^{e_s}$ is the canonical prime factorization of $n \in \mathbb{N}$ then

$$(26) \qquad \beta(n) = p_1 p_2 \cdots p_s, \quad (\beta(1) = 1).$$

2. Similarly as in the above proof one may show that

$$(27) \qquad \frac{(na_1)!\ldots(na_n)!}{n^n(a_1!\ldots a_n!)^n} \in \mathbb{N}.$$

However, (27) is a special case of the more general fact that

(28) $$\frac{(na_1)!\dots(na_n)!}{(n!)^n(a_1!\dots a_n!)^n} \in \mathbb{N}.$$

Indeed, (28) follows from an observation made by WEILL (c.f. [1;p.57]), namely that

(29) $$\frac{(na)!}{n!(a!)^n} \in \mathbb{N}.$$

During the preparation of this note R. TIJDEMAN suggested the following proof of (29),(compare [1;p.57]).

If k and a are positive integers then

(30) $$a! \mid (ka-1)(ka-2)\dots(ka-a)$$

and

(31) $$a! \mid ka(ka-1)\dots(ka-a+1)$$

so that (by subtraction)

(32) $$a! \mid (ka-1)\dots(ka-a+1)\{ka - (ka-a)\}$$

and hence

(33) $$a!k \mid ka(ka-1)\dots(ka-a+1) = \frac{(ka)!}{((k-1)a)!} .$$

From this it is clear that

(34) $$(a!)^n\, n! = \prod_{k=1}^{n} (a!k) \mid (na)! .$$

REFERENCES

[1] BACHMANN, P., *Niedere Zahlentheorie I*, *Leipzig*, 1902.

[2] RUDERMAN, H.D., *Advanced problem* 6121, *American Math. Monthly*, *Vol.* 83, *No.* 10 (1976).